Tutorial 7: Sequences and Series

7.1 Sequences

1. Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

(a) $\{2, 5, 8, 11, ...\}$

(b) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...\right\}$

(c) $\left\{-1, -\frac{1}{3}, \frac{3}{5}, -\frac{5}{7}, ...\right\}$

(d) $\{1, 0, 1, 0, 1, ...\}$

(e) $\left\{0, \frac{1}{2}, 0, \frac{1}{2}, 0, ...\right\}$

2. Does the sequence $\left\{\frac{\cos n}{n}\right\}$ converge or diverge? Find the limit if it is a convergent sequence.

3. Determine whether the given sequence is convergent. If the sequence is convergent, find its limit.

(a) $\left\{\frac{3n^4}{2n^4+1}\right\}$ (b) $\left\{\left(\frac{1}{2}\right)^n + \frac{1}{\sqrt{3^n}}\right\}$ (c) $\left\{\frac{n^2+n+8}{4n^3+n^2}\right\}$

(d) $\left\{\frac{1+n^4}{n^3+3000}\right\}$ (e) $\left\{\frac{\ln n}{n^{\frac{1}{n}}}\right\}$ (f) $\left\{\sin\left(\frac{\pi}{2}+\frac{1}{n}\right)\right\}$

(g) $\left\{\frac{\sin^2 n}{\sqrt{n}}\right\}$ (h) $\left\{\sqrt[n]{10n}\right\}$ (i) $\left\{\frac{n!}{(n+2)!}\right\}$

(j) $\left\{\left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}\right\}$ (k) $\left\{n - \sqrt{n^2-n}\right\}$ (l) $\left\{\frac{n}{5+\sqrt{n}}\right\}$

[Note: (i) Knowing $\lim_{n\to\infty} \sqrt[n]{n} = 1$ may be quite useful.

(ii) The following property may be useful for (f):
If $f(x)$ is a continuous function and the sequence $a_n \to L$, then $f(a_n) \to f(L)$.]

7.2 Series

1. Write down the first four terms of the following series. Does the series converge or diverge? Find the sum if it converges. Explain.

$$\sum_{n=1}^{\infty}\frac{2}{10^n}$$

2. The series $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+...$ can be written in the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ for a suitable value of p. What is the value of p here? Is this series convergent or divergent?

3. Determine whether the given series is convergent or divergent.(Try to use the properties of geometric series or p-series, or the divergence test,) For series that converges, find its sum.

(a) $\frac{1}{4}+\frac{1}{2}+1+2+...$

(b) $3+0.3+0.03+0.003+...$

(c) $2-\frac{4}{3}+\frac{8}{9}-\frac{16}{27}+...$

(d) $1+\frac{2}{\sqrt{2}}+\frac{3}{\sqrt{3}}+\frac{4}{\sqrt{4}}+...$

(e) $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+...$

(f) $\sum_{n=1}^{\infty}\frac{2}{10^n}$

(g) $\sum_{n=0}^{\infty}\left(\frac{2^n-1}{3^n}\right)$

(h) $\sum_{n=1}^{\infty}\frac{3n^2}{5n^2+1}$

(i) $\sum_{n=1}^{\infty}(-1)^n$

(j) $\sum_{n=1}^{\infty}\frac{n^2}{(n+1)(n+3)}$

(k) $\sum_{n=0}^{\infty}\frac{1}{(\sqrt{2})^n}$

(l) $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$

(m) $\sum_{n=1}^{\infty}e^{-n}$

4. Determine whether the given series is convergent or divergent using Comparison Test.

(a) $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$

(b) $\sum_{n=1}^{\infty}\frac{n^2}{n^4+1}$

(c) $\sum_{n=1}^{\infty}\frac{1}{1000n+1}$

(d) $\sum_{n=1}^{\infty}\left((1+\frac{1}{n}\right)e^{-n}$

(e) $\sum_{n=1}^{\infty}\frac{n+2}{n\sqrt{n}}$

(f) $\sum_{n=1}^{\infty}\frac{n+2}{n^2\sqrt{n}}$

(g) $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}+\sqrt[3]{n}}$ (h) $\sum_{n=1}^{\infty}\frac{3-(-1)^n}{n\sqrt{n}}$

(i) $\sum_{n=1}^{\infty}\frac{1+2^n}{1+3^n}$ (j) $\sum_{n=1}^{\infty}\frac{1+5^n}{1+3^n}$

5. Use the ratio test or the root test to decide on the convergence of the series $\sum_{n=0}^{\infty} a_n$.

(a) $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$ (b) $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$ (c) $\sum_{k=1}^{\infty}\frac{k!}{k^k}$

(d) $\sum_{n=1}^{\infty}\frac{10^n}{n^n}$ (e) $\sum_{n=1}^{\infty}\frac{10^n}{n!}$ (f) $\sum_{n=1}^{\infty}\frac{10^n}{n\cdot 4^{2n}}$

(g) $\sum_{n=1}^{\infty}\frac{10^n}{n\cdot 3^{2n}}$ (h) $\sum_{n=1}^{\infty}\left(\frac{2n+5}{4n+1}\right)^n$ (i) $\sum_{n=1}^{\infty}\left(\frac{4n+1}{2n+9}\right)^n$

7.3 Taylor Series and Maclaurin series

1. Find the Taylor series for $f(x)$ at the given value of a.

(a) $f(x)=3x^2+2x+1$, $a=3$

(b) $f(x)=e^{2x}\sin x$, $a=\frac{\pi}{2}$

(c) $f(x)=\frac{1}{x}$, $a=1$

(d) $f(x)=\frac{1}{3-x}$, $a=2$

2. Find the Maclaurin series for $f(x)$.

(a) $f(x)=\frac{1}{1-x}$

(b) $f(x)=\cos 3x$

(c) $f(x)=xe^x$

7.4 Fourier Series

In each of the following, a periodic function of period 2π is specified over one period.

(i) Sketch a graph of the function for $-2\pi < x < 2\pi$, $-3\pi < x < 3\pi$ and $-4\pi < x < 4\pi$.

(ii)Obtain a Fourier series representation for the function.

1, $f(x) = \begin{cases} 1 & \text{if } -\pi \le x < 0 \\ -1 & \text{if } 0 \le x < \pi \end{cases}$

2. $f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ \cos x & \text{if } 0 \le x < \pi \end{cases}$

4. $f(x) = \begin{cases} -1 & \text{if } -\pi \le x < -\frac{\pi}{2} \\ 1 & \text{if } -\frac{\pi}{2} \le x < 0 \\ 0 & \text{if } 0 \le x < \pi \end{cases}$

5. $f(x) = x$ for $-\pi \le x < \pi$

6. $f(x) = x^2$ for $-\pi \le x < \pi$